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Resonant Amplification of Sound by Conduction Electrons*

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A Boltzmann equation technique is used to calculate the angular distribution of sound amplified by conduction electrons under various conditions. It was found that if $\mathbf{v}_d$ is the drift velocity of the electrons, and $\hat{q}$ a unit vector in the direction of propagation of sound, amplification occurs for all $\hat{q}$ such that $\hat{q}\cdot\mathbf{v}_d>v_s$, where v_s is the velocity of sound. Resonance peaks in the amplification occur at certain directions of propagation. For a semimetal in crossed electric and magnetic fields resonant amplification occurs when $\hat{q}\cdot\mathbf{v}_d-v_F(\hat{q}\cdot\hat{H})\approx v_s$ (where $\hat{H}$ is a unit vector in the direction of the magnetic field, and v_F the Fermi velocity) if $ql\gg1$, where l is the mean free path of the electrons and q the wave number of the sound wave; if $ql\ll1$ resonance peaks occur for $\hat{q}\cdot\mathbf{v}_d\approx v_s$. The peak amplification is rather insensitive to the value of v_d/v_s, and varies with frequency as ql. For a metal or semiconductor in an applied electric field only, resonant amplification occurs for those directions of propagation such that $\hat{q}\cdot\mathbf{v}_d\approx v_s$, if $ql\ll1$, and the peak amplification is independent of v_d/v_s. There are no resonances in this case of $ql\gg1$.

I. INTRODUCTION

RECENT experiments strongly imply that whenever the drift velocity of electrons exceeds the velocity of sound, amplification of sound will result. In particular, Hutson, McFee, and White[1] and Wang[2] have observed amplification in CdS in the presence of a static electric field. It has also been observed that the resistance of a metal changes when the drift velocity exceeds the velocity of sound. This change in resistance, which was observed in piezoelectric semiconductors by Smith[3] and McFee,[4] and in bismuth in the presence of a strong magnetic field by Esaki,[5] has been attributed to the creation of phonons which occurs for drift velocities greater than the velocity of sound.

Several theoretical calculations of the amplification of sound have been made, but these were limited to the special condition that the direction of propagation of sound is in the direction of the drift velocity. Hutson, McFee, and White[1] gave a phenomenological calculation for a semiconductor in a static electric field; and Spector[6] used a Boltzmann equation technique for the same problem. Dumke and Haering,[7] and Hopfield[8] have given phenomenological treatments of the amplification of sound in a semimetal in the presence of crossed electric and magnetic fields. Although the special geometry considered by these authors is appropriate to an amplification/attenuation experiment, in which the direction of propagation of sound may be chosen to coincide with the direction of the drift velocity, it is not appropriate to a resistance experiment, for in the latter case, no external sound wave is impressed: Sound is produced by the electrons, but is not, of course, necessarily in the direction of the drift velocity. In order to calculate the effects of phonon creation upon the resistance, the spectrum of the phonons must be known, both with respect to frequency and angular distribution. In the present paper this spectrum is investigated in some generality, both for semiconductors in an external electric field, and for semimetals in crossed electric and magnetic fields. The method used is a Boltzmann equation technique, which is applicable both to the frequency range, $ql\ll1$ and $ql\gg1$, and, therefore, the results of Dumke and Haering which were applicable only to the case $ql\ll1$, are also generalized in this sense.

It is found that there are preferred angles for sound production, at which there are strong resonances in the amplification. As a by-product of this investigation, the existence of these favorable directions should prove to be useful in a search for the as yet unobserved amplification of sound in semimetals in crossed electric and magnetic fields.

In this section an examination is made of the general conditions which are necessary for amplification of sound to occur. Simple physical arguments will also be given for the resonances in the angular distribution of the amplified sound.

In attempting to understand sound amplification one is immediately struck by the analogy of Čerenkov radiation of light. If a free electron has a velocity greater than the velocity of light in a medium, photons are created; if its velocity is greater than the velocity of sound, one would expect phonons to be created. The analogy is not complete, however, for in this case we are not dealing with free electrons, but with a distribution of electrons within a metal. Also, the Fermi velocity is typically several orders of magnitude greater than the velocity of sound, and it is difficult to see why a small added drift velocity can radically change the behavior of the metal, from an attenuator of sound to an amplifier. Furthermore the drift velocity is a statistical average velocity, but it is the velocity of the

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[1] A. R. Hutson, J. H. McFee, and D. L. White, Phys. Rev. Letters **7**, 237 (1961).
[2] W. C. Wang, Phys. Rev. Letters **9**, 443 (1962).
[3] R. W. Smith, Phys. Rev. Letters **9**, 87 (1962).
[4] J. H. McFee, J. Appl. Phys. **34**, 1548 (1963).
[5] L. Esaki, Phys. Rev. Letters **8**, 4 (1962).
[6] H. N. Spector, Phys. Rev. **127**, 1084 (1962).
[7] W. P. Dumke and R. R. Haering, Phys. Rev. **126**, 1974 (1962).
[8] J. J. Hopfield, Phys. Rev. Letters **8**, 311 (1962).

individual electron which is important in each electron-phonon interaction.

In spite of these objections closer examination reveals that the analogy of Čerenkov radiation is quite useful; the remainder of this section will be devoted to a qualitative description of sound amplification in terms of this analogy.

For a Fermi distribution, the exclusion principle is responsible for the fact that no amplification occurs under an equilibrium distribution; although very high velocities exist within the metal, no phonons can be created (at 0°K) because all electronic states of lower energy are occupied. But suppose that by some mechanism the distribution becomes skewed by a velocity vector $\mathbf{V}$, so that all states are occupied for $(\mathbf{v}-\mathbf{V})^2 \leqslant v_F^2$, and none occupied for $(\mathbf{v}-\mathbf{V})^2 > v_F^2$. Then phonon creation may take place if $\mathbf{V}$ satisfies a certain condition; this condition will now be found.

The initial velocity $\mathbf{v}_i$ and the final velocity $\mathbf{v}_f$ of the electron must satisfy the inequalities

$$(\mathbf{v}_i-\mathbf{V})^2 \leqslant v_F^2,\qquad (\mathbf{v}_f-\mathbf{V})^2 \geqslant v_F^2, \tag{1.1}$$

so that

$$(\mathbf{v}_i-\mathbf{V})^2 \leqslant (\mathbf{v}_f-\mathbf{V})^2. \tag{1.2}$$

The initial and final velocities are related by energy-momentum conservation:

$$\tfrac{1}{2}mv_i^2=\tfrac{1}{2}mv_f^2+\hbar\omega, \tag{1.3a}$$

$$\mathbf{v}_{i\perp}=\mathbf{v}_{f\perp}, \tag{1.3b}$$

$$m\mathbf{v}_{i\parallel}=m\mathbf{v}_{f\parallel}+\hbar\mathbf{q}. \tag{1.3c}$$

In these expressions $\parallel$ and $\perp$ refer to the velocity components parallel and perpendicular to the direction of propagation of the phonon; also, $\mathbf{q}$ and ω are the wave vector and frequency of the phonon. Rewriting these expressions in a form slightly more convenient for our purposes:

$$\mathbf{v}_i\cdot\hat{q}=v_s+\hbar q/2m,\qquad \mathbf{v}_f\cdot\hat{q}=v_s-\hbar q/2m, \tag{1.4}$$

where $\hat{q}$ is a unit vector in the direction of propagation of sound, and v_s the velocity of sound. In this form the energy-momentum conservation relations are the conditions for Čerenkov radiation. They state that phonons may be created by an individual electron only if $v_i > v_s$; and, furthermore, the direction of the radiation is related to the direction of the initial velocity in such a way that phonons are radiated only *on* the cone around the direction of the initial velocity whose angle θ satisfies

$$\cos\theta=v_s/v_i. \tag{1.5}$$

In this expression $\hbar q/2m$ has been neglected in comparison with v_s. In fact $\hbar q/2mv_s \approx 4\times10^{-10}\nu$, where ν is given in cycles/sec, so that it is almost always negligible.

When the relationships (1.4) and (1.3b) are substituted in (1.2) a straightforward manipulation gives the condition which $\mathbf{V}$ must satisfy in order that phonons may be created:

$$\mathbf{V}\cdot\hat{q} \geqslant v_s. \tag{1.6}$$

These arguments may easily be extended to Fermi distributions at finite temperature, and, thus, to semiconductors. For an unskewed distribution, the electron population decreases monotonically with increasing velocity, so that there are more empty states available for the final state electron with $v_f \geqslant v_i$ than with $v_f \leqslant v_i$. Consequently, absorption of phonons will be more probable than emission (since the transition probability for absorption of a phonon by one electron is equal to that for emission) and the net result will be attenuation. Similarly, in a skewed distribution there are more final states available with $(\mathbf{v}_f-\mathbf{V})^2 \geqslant (\mathbf{v}_i-\mathbf{V})^2$ than with $(\mathbf{v}_f-\mathbf{V})^2 \leqslant (\mathbf{v}_i-\mathbf{V})^2$. Use of (1.4) and (1.3b) (together with the corresponding equations for phonon absorption) shows that emission of phonons is more probable than absorption of phonons only if $\mathbf{V}\cdot\hat{q} \geqslant v_s$, so that the condition for amplification is the same as (1.6).

The physical significance of $\mathbf{V}$ is obvious: If a distribution is skewed by a vector $\mathbf{V}$, this means that the distribution function $f(\mathbf{v})=f_0(\mathbf{v}-\mathbf{V})$ where f_0 is the distribution in the absence of the skewing mechanism. Therefore

$$\mathbf{V}=\int \mathbf{v}f(\mathbf{v})d^3v=\langle\mathbf{v}\rangle, \tag{1.7}$$

where $\langle\mathbf{v}\rangle$ is the average velocity of the distribution. For crossed electric and magnetic fields

$$\mathbf{V}=\langle\mathbf{v}\rangle=\frac{eE\tau}{m}\frac{(\hat{n}_1+\omega_c\tau\hat{n}_2)}{1+(\omega_c\tau)^2}=\frac{\mathbf{v}_d^E+(\omega_c\tau)^2\mathbf{v}_d^H}{1+(\omega_c\tau)^2}, \tag{1.8}$$

where $\hat{n}_1$ is a unit vector in the direction of the electric field and $\hat{n}_2$ is a unit vector in the direction $\mathbf{E}\times\mathbf{H}$; ω_c is the electron cyclotron frequency eH/mc; $\mathbf{v}_d^E$ is the electron drift velocity $e\mathbf{E}\tau/m$ and $\mathbf{v}_d^H$ is the drift velocity $c(\mathbf{E}\times\mathbf{H})/H^2$.

Fermi statistics allow creation of phonons for $V > v_s$, which in this case means

$$\frac{\omega_c\tau v_d^H}{[1+(\omega_c\tau)^2]^{1/2}}\equiv\frac{v_d^E}{[1+(\omega_c\tau)^2]^{1/2}}>v_s. \tag{1.9}$$

For low magnetic fields ($\omega_c\tau \ll 1$) inequality (1.6) becomes

$$\mathbf{v}_d^E\cdot\hat{q}>v_s \tag{1.10}$$

and for high magnetic fields ($\omega_c\tau \gg 1$) it is

$$\mathbf{v}_d^H\cdot\hat{q}>v_s. \tag{1.11}$$

The angular distribution of radiated sound will now be investigated qualitatively. For this purpose the analogy of Čerenkov radiation is extremely helpful, for condition (1.5) gives an extremely sharp correlation between the direction of the phonon and the initial velocity of the electron. If $\langle \mathbf{v}\rangle_\tau$ is the initial velocity of an electron within the metal, averaged over a suitable time of interaction, then because of (1.5) the amplification of sound due to this electron would be expected to be a function with a resonance denominator in $(\langle \mathbf{v}\rangle_\tau\cdot\hat{q}-v_s)$.[9] This is a restatement of energy-momentum conservation, which is based on the particle description of sound (phonons). Another argument which gives the same result is based on the wave description of sound: In order to have resonant transfer of energy between an electron and the sound wave, the electron must move in phase with the sound wave. If $\mathbf{v}$ is the velocity of the electron it will stay in phase with the sound wave only if $\mathbf{v}\cdot\hat{q}=v_s$, which is the same condition as (1.5).

This latter argument suggests that a suitable time of interaction is the time required for an electron to move a wavelength, i.e., $\approx 1/qv_F$. If this is the case, one would expect radically different behavior for $ql\ll 1$ and $ql\gg 1$, where l is the electron mean free path.

First consider $ql\ll 1$. Then many collisions take place while the electron is moving a wavelength, so that the average electron velocity during the time of interaction is $\langle \mathbf{v}\rangle_\tau=\mathbf{v}_d{}^E=e\mathbf{E}\tau/m$ or $\langle \mathbf{v}\rangle_\tau=\mathbf{v}_d{}^H=c\mathbf{E}\times\mathbf{H}/H^2$, depending upon the case under consideration. Therefore, we expect resonant amplification for $\langle \mathbf{v}\rangle_\tau\cdot\hat{q}=\mathbf{v}_d\cdot\hat{q}=v_s$. This means that most of the amplification will be very close to the Čerenkov cone defined by $\cos\theta=v_s/v_d$; it will be within the cone because of the necessary conditions upon the electron distribution (1.10) and (1.11). Outside the cone (and very close to it) there will be resonant *attenuation* of sound, because, as we have seen, the energy-momentum condition for annihilation of phonons is almost the same as for their creation.

On the other hand, if $ql\gg 1$, the electron is moving solely under the influence of the external fields during the time of interaction, because it does not make any collisions during this time. Its velocity at time t is found by solving the equations of motion for a free electron moving under the influence of the applied fields, and in the case of an applied electric field only is given by:

$$\mathbf{v}(t)=\mathbf{v}(0)+\mathbf{v}_d{}^E(t/\tau). \qquad (1.12)$$

Now $v(0)\approx v_F$ for the majority of electrons, and, furthermore, $t/\tau<1/qv_F\tau\ll 1$, so that the term in $\mathbf{v}_d{}^E$ may be neglected in comparison with $\mathbf{v}(0)$. Thus, the average velocity during this time is $\mathbf{v}(0)\equiv\mathbf{v}$. The contribution to amplification due to a specific electron will then have a resonance denominator in $(\hat{q}\cdot\mathbf{v}-v_s)$. When this function is integrated over the entire distribution a function with resonance behavior in (v_F-v_s) will result. However, $v_F\gg v_s$, so that the resonance condition cannot be satisfied, and therefore resonant amplification of sound is not expected for $ql\gg 1$, in the case of an applied electric field only.

[9] For attenuation of sound we expect the same resonance denominator because energy-momentum conservation for creation *or* destruction of phonons is $v\cdot\hat{q}=v_s\pm\hbar q/2m\approx v_s$ in both cases.

If $ql\gg 1$ and crossed electric and magnetic fields are applied, the velocity of an electron at time t is given by

$$\mathbf{v}(t)=\mathbf{v}_d{}^H+v_H\hat{H}+\begin{pmatrix}\cos\omega_c t & -\sin\omega_c t & 0\\ \sin\omega_c t & \cos\omega_c t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}v_x(0)-v_d{}^H\\ v_y(0)\\ 0\end{pmatrix}, \qquad (1.13)$$

where v_H is the (constant) projection of the velocity in the direction of the magnetic field and $\hat{H}$ is a unit vector in the direction of the magnetic field. For high magnetic fields ($\omega_c\gg qv_F$) the circular part of the velocity averages to zero during the interaction time, so that the contribution to amplification of sound for one electron is expected to have a resonance denominator in

$$(\langle \mathbf{v}\rangle_\tau\cdot\hat{q}-v_s)=(\mathbf{v}_d{}^H\cdot\hat{q}+v_H\hat{H}\cdot\hat{q}-v_s). \qquad (1.14)$$

Integration over the entire distribution will result in a sum of functions with resonance behavior as functions of

$$(\mathbf{v}_d{}^H\cdot\hat{q}\pm v_F\hat{H}\cdot\hat{q}-v_s). \qquad (1.15)$$

Resonant peaks in the amplification are, therefore, to be expected at those directions of propagation $\hat{q}$ for which expressions (1.15) vanish, if, in addition, the requirement (1.11) $\mathbf{v}_d{}^H\cdot\hat{q}\geqslant v_s$ is met. If this latter condition is not fulfilled, resonant attenuation is expected at these directions. For $v_d=0$ this effect (the tilt effect) was observed,[10] and was calculated theoretically by Spector.[11]

The remainder of this paper will give a quantitative treatment of these effects based on a Boltzmann equation technique developed by Cohen, Harrison and Harrison[12] for the attenuation of sound. The qualitative features discussed above will be shown to be correct quantitatively. In Sec. II the generalization to all directions of propagation of the work of Spector[6] for the case in which an electric field only is applied will be presented. In Sec. III, the extension of the formalism of Cohen, Harrison, and Harrison to the case of crossed electric and magnetic fields will be given; the results for this case will be discussed in Sec. IV.

II. AMPLIFICATION IN APPLIED ELECTRIC FIELD

The angular distribution of sound amplified by interaction with conduction electrons in the presence of a static electric field will be discussed in this section. The formalism for this problem was treated fully by Spector,[6] but he considered only the special cases in which the direction of propagation of sound is either parallel or perpendicular to the drift velocity. The reader is referred to his paper for the derivation of the formalism, notation, etc. We shall confine ourselves to the case in which the interaction between the electrons and sound wave is mainly due to the deformation potential, i.e.,

[10] D. H. Reneker, Phys. Rev. **115**, 303 (1959).
[11] H. N. Spector, Phys. Rev. **120**, 1261 (1960).
[12] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).

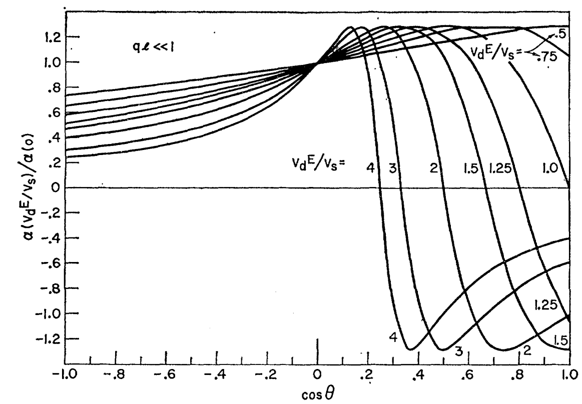

FIG. 1. Electric field only: Attenuation (relative to zero-field attenuation) as a function of the angle between the direction of propagation of sound and the drift velocity, for $ql\ll1$, and several values of v_d^E/v_s. The parameter C was found by fitting (2.2) to the experimental Fig. 2, Curve B of Hutson *et al.* (Ref. 1) (ν=45 Mc/sec).

$(v_F/v_s)^2(\omega/\omega_p)^2\mathbf{C}_{ij}/mv_F^2\gg1$. In this case using Eqs. (2.13b) and (2.14) of Spector's paper, we find that the attenuation of waves polarized in the jth direction is given by:

$$\alpha_j=\frac{N_0m}{\rho v_s\tau}\left(\frac{q^2\mathbf{C}_{jz}}{m\omega_p^2}\right)^2\operatorname{Re}\frac{1}{(\sigma_{zz}'+\Sigma_{zz}'-i\gamma)},\tag{2.1}$$

where the direction of propagation of sound $\hat{q}$ is in the z direction. Use of Spector's equations (2.7) show that σ_{zz}' and Σ_{zz}' are the same for arbitrary direction of propagation as those given by Spector's equations (3.1a) and (3.1b) except that μ is replaced by $\bar{\mu}$, where $\bar{\mu}=\mathbf{v}_d^E\cdot\hat{q}/v_s$. Therefore, Spector's results (3.5b, c) and (3.6b, c) for the attenuation remain unchanged, except for the substitution of $\bar{\mu}$ for μ. Upon making this substitution we obtain

$$\alpha_j=\kappa(ql)^2\frac{(v_s-\hat{q}\cdot\mathbf{v}_d^E)}{(v_s-\hat{q}\cdot\mathbf{v}_d^E)^2+C^2},\quad\text{for}\quad ql\ll1\tag{2.2}$$

$$\alpha_j=\frac{\pi}{6}\kappa(ql)^3\frac{(v_s-\hat{q}\cdot\mathbf{v}_d^E)}{C^2},\quad\text{for}\quad ql\gg1,\tag{2.3}$$

where

$$\kappa=\frac{N_0(\mathbf{C}_{zj})^2}{\rho v_s^2\tau mv_F^2}\quad\text{and}\quad C=v_s\frac{(\omega_p\tau)^2}{\omega\tau}\left[1+\frac{1}{3}\left(\frac{v_F}{v_s}\right)^2\left(\frac{\omega}{\omega_p}\right)^2\right].$$

As predicted by the qualitative arguments of the introduction, amplification occurs (the attenuation becomes negative) when $\hat{q}\cdot\mathbf{v}_d^E\geqslant v_s$. Also, there is resonant amplification (and attenuation) for $ql\ll1$, and no resonances for $ql\gg1$. The maximum of the resonances for $ql\ll1$ occur for those angles for which $\hat{q}\cdot\mathbf{v}_d^E=v_s\pm C$, and the peak amplification is equal to $\kappa(ql)^2/2C$, independent of v_d^E/v_s, provided that $v_d^E\geqslant v_s+C$.[13] This point should be emphasized, for from previous calculations it might erroneously have been inferred that the amplification decreases as the drift velocity is increased, but this is of course true only at each angle; if v_d^E/v_s is large enough, there is always an angle at which the amplification attains the same maximum, independent of v_d^E/v_s.

Although the expression for the attenuation (2.2) contains a resonance denominator as expected, the position of the peak amplification and the width of the peak depends strongly on the damping factor C. This factor is not particularly small for cases of physical interest, such as CdS, so that we expect broad peaks shifted quite a bit from the Čerenkov angle. An examination of the damping factor shows that the plasma oscillations contribute strongly to the damping. Figure 1 is a plot of the attenuation as a function of the angle θ between the direction of propagation and the drift velocity for several values of v_d^E/v_s, calculated for the physical conditions appropriate to the experiments of Hutson *et al.*[1]

For $ql\gg1$, no resonant amplification occurs; amplification occurs within the cone $\cos\theta\geqslant v_s/v_d^E$; it increases linearly with $\cos\theta$, and also increases linearly with v_d^E/v_s. Plots of the amplification as a function of θ for several v_d^E/v_s are given in Fig. 2.

III. FORMAL THEORY. SOUND ATTENUATION BY SEMIMETALS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

A. The Constitutive Equation

The method which will be used to find the amplification was developed by Cohen, Harrison, and Harrison[12] and Harrison.[14] This method consists of the simultaneous solution of Boltzmann's equations for the electronic carriers with Maxwell's equations for a self-consistent field which accompanies the sound wave. The power dissipated from the sound wave is then calculated, and from this quantity the attenuation or amplification is found.

As a model of a semimetal we consider a system containing N_0 electrons and N_0 holes per unit volume mov-

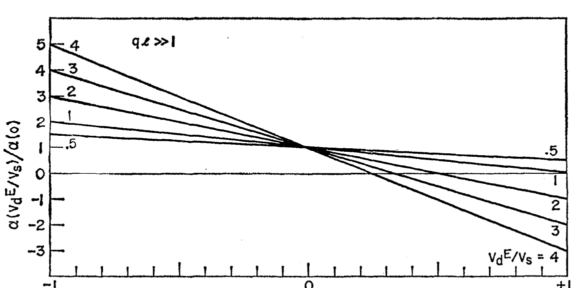

FIG. 2. Electric field only: Attenuation (relative to zero-field attenuation) as a function of the angle between the direction of propagation of sound and the drift velocity, for $ql\gg1$, and several values of v_d^E/v_s.

[13] For $v_s\leqslant v_d^E\leqslant v_s+C$ there is no angle at which this maximum amplification, $\kappa(ql)^2/2C$ is attained, and the amplification increases monotonically towards the forward direction.

[14] M. J. Harrison, Phys. Rev. **119**, 1260 (1960).

ing through a uniform neutral background. A sound wave of propagation vector $\mathbf{q}$ and frequency ω manifests itself as a velocity field $u(\mathbf{r},t)\propto\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$ in the neutral background. Interactions between the particles are replaced by interactions of the individual particles with a self-consistent electric field derived from Maxwell's equations. Interaction of the particles with the sound wave is represented partially by interaction with this self-consistent field, and partially by interaction with a deformation potential proportional to the local dilatation of the lattice. The effective mass approximation is used, and both electrons and holes are assumed to have isotropic effective masses.

Maxwell's equations for the self-consistent electromagnetic field may be written as

$$\mathbf{j}_s=-\mathbf{B}\cdot\boldsymbol{\mathcal{E}}_s. \tag{3.1}$$

Here $\mathbf{j}_s$ and $\boldsymbol{\mathcal{E}}_s$ are those components of the current and field which vary, like the velocity field, as $\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$. $\mathbf{B}$ is the tensor

$$\mathbf{B}=i[\beta\mathbf{I}-(\beta+\gamma)\hat{q}\hat{q}], \tag{3.2}$$

where $\beta=(c/v_s)^2(\omega/4\pi)[1-\epsilon(v_s/c)^2]$ and $\gamma=\epsilon\omega/4\pi$; $\hat{q}$ is a unit vector in the direction of propagation of sound, ϵ the dielectric constant of the neutral background, and v_s the velocity of sound.

The current $\mathbf{j}_s$ is the sum of the ac part of the electron and hole currents. These currents are obtained from the distribution functions which are the solutions of Boltzmann's equations:

$$\begin{aligned}\mathbf{j}_e(\mathbf{r},t)&=-e\int\mathbf{v}f_e(\mathbf{r},\mathbf{v},t)d^3v,\\ \mathbf{j}_h(\mathbf{r},t)&=+e\int\mathbf{v}f_h(\mathbf{r},\mathbf{v},t)d^3v.\end{aligned} \tag{3.3}$$

Equation (3.1) is essentially a relationship between f_e, f_h, and $\boldsymbol{\mathcal{E}}_s$. Two other relations between these quantities are Boltzmann's equations:

$$\begin{aligned}\frac{\partial f_e}{\partial t}+\mathbf{v}\cdot\frac{\partial f_e}{\partial\mathbf{r}}+\frac{\mathbf{F}_e}{m_e}\cdot\frac{\partial f_e}{\partial\mathbf{v}}&=\left.\frac{\partial f_e}{\partial t}\right|_{\text{coll}},\\ \frac{\partial f_h}{\partial t}+\mathbf{v}\cdot\frac{\partial f_h}{\partial\mathbf{r}}+\frac{\mathbf{F}_h}{m_h}\cdot\frac{\partial f_h}{\partial\mathbf{v}}&=\left.\frac{\partial f_h}{\partial t}\right|_{\text{coll}}.\end{aligned} \tag{3.4}$$

In these equations the forces $\mathbf{F}_e$ and $\mathbf{F}_h$ which act upon the electrons and holes are the sums of the Lorentz force and the deformation potential force:

$$\begin{aligned}\mathbf{F}_e&=-e\left[\boldsymbol{\mathcal{E}}+\frac{\mathbf{v}}{c}\times\boldsymbol{\mathcal{H}}+\frac{\mathbf{q}\mathbf{q}\mathbf{C}_e}{ie\omega}\mathbf{u}\right]\\ \mathbf{F}_h&=+e\left[\boldsymbol{\mathcal{E}}+\frac{\mathbf{v}}{c}\times\boldsymbol{\mathcal{H}}-\frac{\mathbf{q}\mathbf{q}\mathbf{C}_h}{ie\omega}\mathbf{u}\right].\end{aligned} \tag{3.5}$$

Here $\mathbf{C}_e$ and $\mathbf{C}_h$ are the deformation potential tensors. The field $\boldsymbol{\mathcal{E}}$ is the sum of the external static field $\mathbf{E}_0$, the Hall field $\mathbf{E}_H$, and the self-consistent electric field $\boldsymbol{\mathcal{E}}_s$. Similarly, $\boldsymbol{\mathcal{H}}$ is the sum of the external static field $\mathbf{H}$ and the self-consistent field $\boldsymbol{\mathcal{H}}_s$. In the case under consideration the external fields are perpendicular to each other: $\mathbf{E}_0\perp\mathbf{H}$. Therefore, $\mathbf{E}_H$ and also the resultant $\mathbf{E}=\mathbf{E}_0+\mathbf{E}_H$ are perpendicular to $\mathbf{H}$. The coordinate system which will be used is the following: the z axis in the direction of $\mathbf{H}$, and the y axis in the direction of the resultant $\mathbf{E}$. For such a geometry a free electron or hole will develop a drift velocity in the direction of the x axis with the magnitude $\mathbf{v}_d{}^H=c\mathbf{E}\times\mathbf{H}/H^2$.

The collision terms of the Boltzmann equation will be treated by the use of the relaxation time ansatz. Recombination collisions will be neglected here, so that the results will be strictly correct only for infinite recombination time. However, recombination collisions are not expected to change the qualitative features of our results.[15]

Bearing these assumptions in mind, the collision terms of the Boltzmann equations are

$$\begin{aligned}\left.\frac{\partial f_e}{\partial t}\right|_{\text{coll}}&=-\frac{f_e-f_{\text{se}}}{\tau_e},\\ \left.\frac{\partial f_h}{\partial t}\right|_{\text{coll}}&=-\frac{f_h-f_{\text{sh}}}{\tau_h}.\end{aligned} \tag{3.6}$$

A scattering collision will cause the distribution to relax towards the equilibrium distribution centered about the impurity velocity $\mathbf{u}$. Also, scattering is local and cannot change the electron (hole) density. Therefore,

$$\begin{aligned}f_{\text{se}}(\mathbf{r},\mathbf{v},t)&=f_0[\mathbf{v}-\mathbf{u}(\mathbf{r},t),E_F{}^e(\mathbf{r},t)],\\ f_{\text{sh}}(\mathbf{r},\mathbf{v},t)&=f_0[\mathbf{v}-\mathbf{u}(\mathbf{r},t),E_F{}^h(\mathbf{r},t)],\end{aligned} \tag{3.7}$$

where $f_0(\mathbf{v},E_F)$ is the equilibrium Fermi distribution, and $E_F{}^e(\mathbf{r},t)$ and $E_F{}^h(\mathbf{r},t)$ are chosen to give the local electron and hole densities $N_e(\mathbf{r},t)$ and $N_h(\mathbf{r},t)$.

Boltzmann's equations are solved by a method due to Chambers.[16] This solution is

$$f(\mathbf{r},\mathbf{v},t)=\int_{-\infty}^{t}f_s(\mathbf{r}',\mathbf{v}',t')e^{-(t-t')/\tau}\frac{dt'}{\tau}. \tag{3.8}$$

The distribution function will be expanded to the first order in $\mathbf{u}$ and quantities proportional to $\mathbf{u}$; and also

[15] The effect of recombination collisions may be estimated by examining the attenuation of sound in the presence of a magnetic field only. Recombination collisions are expected to broaden the resonance peaks at $(\hat{q}\cdot\hat{H})=v_s/v_F$. However, the observed resonance peaks (Ref. 10) are so narrow that they are consistent with the formula calculated neglecting recombination (Ref. 11). It is, therefore, felt that it is reasonable to neglect recombination in the case of applied electric and magnetic fields also.

[16] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1962); **A238**, 344 (1957).

to first order in the static electric field $\mathbf{E}$. It is, therefore, necessary to expand to the second order in $\boldsymbol{\mathcal{E}}$, because all bilinear terms in $\mathbf{E}$ and $\boldsymbol{\mathcal{E}}_s$ must be retained.

Expansion of $f_s(\mathbf{r},\mathbf{v},t)$ in a Taylor's series results in

$$f(\mathbf{r},\mathbf{v},t)=f_0(\mathbf{v},E_F{}^0)+f_1+f_2, \tag{3.9}$$

where

$$f_1=\frac{\partial f_0}{\partial E}\int_{-\infty}^{t}[\Delta E-m^*\mathbf{v}'\cdot\mathbf{u}'-(E_F{}'-E_F{}^0)]e^{-(t-t')/\tau}\frac{dt'}{\tau}$$

and

$$f_2=\frac{1}{2}\left(\frac{\partial^2 f_0}{\partial E^2}\right)\int_{-\infty}^{t}[\Delta E-m^*\mathbf{v}'\cdot\mathbf{u}'-(E_F{}'-E_F{}^0)]^2e^{-(t-t')/\tau}\frac{dt'}{\tau}. \tag{3.10}$$

In these expressions the primed quantities refer to time t'; $\Delta E=E(\mathbf{v}')-E(\mathbf{v})$; and m^* is equal to either one of the effective masses, m_e or m_h. If the density of the charge carrier is expanded as $N(\mathbf{r},t)=N_0+N_1(\mathbf{r},t)$ and $N_1\ll N_0$, then

$$E_F(\mathbf{r},t)-E_F{}^0=\tfrac{2}{3}E_F{}^0N_1/N_0. \tag{3.11}$$

The following identity will be used to evaluate $f_1(\mathbf{r},\mathbf{v},t)$:

$$\int_{-\infty}^{t}g(t')e^{-(t-t')/\tau}\frac{dt'}{\tau}=g(t)-\tau\int_{-\infty}^{t}\frac{dg}{dt'}e^{-(t-t')/\tau}\frac{dt'}{\tau}. \tag{3.12}$$

Thus,

$$\int_{-\infty}^{t}(\Delta E)e^{-(t-t')/\tau}\frac{dt'}{\tau}=-\bar{e}\tau\int_{-\infty}^{t}\left(\mathbf{E}+\boldsymbol{\mathcal{E}}_s{}'-\frac{\mathbf{qqC}}{i\bar{e}\omega}\mathbf{u}'\right)\cdot\mathbf{v}'e^{-(t-t')/\tau}\frac{dt'}{\tau}, \tag{3.13}$$

where $\bar{e}$ is the charge of either electrons or holes (including the sign of the charge). Further use of identity (3.12) gives

$$-\bar{e}\tau\int_{-\infty}^{t}(\mathbf{E}\cdot\mathbf{v}')e^{-(t-t')/\tau}\frac{dt'}{\tau}=-\bar{e}\mathbf{E}\tau\frac{(v_y+\bar{\omega}_c\tau v_x)}{1+(\bar{\omega}_c\tau)^2}+\frac{\bar{e}\tau(\bar{\omega}_c\tau)v_d{}^H}{1+(\bar{\omega}_c\tau)^2}\int_{-\infty}^{t}\left[\bar{\omega}_c\tau\left(\mathcal{E}_{s,x}{}'-\left[\frac{\mathbf{qqC}}{i\bar{e}\omega}\mathbf{u}'\right]_x\right)+\left(\mathcal{E}_{s,y}{}'-\left[\frac{\mathbf{qqC}}{i\bar{e}\omega}\mathbf{u}'\right]_y\right)\right]e^{-(t-t')/\tau}\frac{dt'}{\tau}, \tag{3.14}$$

where $\bar{\omega}_c=\bar{e}H/m^*c$. Substitution of this result in (3.13) and use of (3.10) and (3.11) results in

$$f_1=\left(-\frac{\partial f_0}{\partial E}\right)\left\{\frac{\bar{e}E\tau(v_y+\bar{\omega}_c\tau v_x)}{1+(\bar{\omega}_c\tau)^2}+\int_{-\infty}^{t}\left[\bar{e}\tau\left(\boldsymbol{\mathcal{E}}_s{}'-\frac{\mathbf{qqCu}'}{i\bar{e}\omega}+\frac{m^*\mathbf{u}'}{\bar{e}\tau}\right)\cdot\mathbf{v}'+\frac{2}{3}\frac{E_F{}^0}{N_0}N_1{}'\right.\right.$$
$$\left.\left.-\frac{\bar{e}\tau(\bar{\omega}_c\tau)v_d{}^H}{1+(\bar{\omega}_c\tau)^2}\left(\bar{\omega}_c\tau\left[\mathcal{E}_{s,x}{}'-\left(\frac{\mathbf{qqCu}'}{i\bar{e}\omega}\right)_x\right]+\left[\mathcal{E}_{s,y}{}'-\left(\frac{\mathbf{qqCu}'}{i\bar{e}\omega}\right)_y\right]\right)\right]e^{-(t-t')/\tau}\frac{dt'}{\tau}\right\}. \tag{3.15}$$

In this expression all terms quadratic in $\mathbf{E}$ or in terms which vary as the sound wave have been neglected. The first term above will contribute only to the dc current, and, therefore, will not contribute to the attenuation of sound. Similarly, terms quadratic in $\mathbf{E}$ which are neglected above do not contribute to sound attenuation.

In the expression for f_2 we again retain only those terms which are linear both in $\mathbf{E}$ and in the quantities which vary as the sound wave. If ΔE is separated into ΔE_s, the energy increment due to the sound wave, and ΔE_0, the increment due to the static electric field ($\Delta E=\Delta E_s+\Delta E_0$), then the only terms which need be retained in f_2 are

$$f_2=\left(\frac{\partial^2 f_0}{\partial E^2}\right)\int_{-\infty}^{t}(\Delta E_0)\left[\Delta E_s-m^*\mathbf{v}'\cdot\mathbf{u}'-\frac{2}{3}\frac{E_F{}^0}{N_0}N_1{}'\right]e^{-(t-t')/\tau}\frac{dt'}{\tau}. \tag{3.16}$$

From the equations of motion of an electron in crossed electric and magnetic fields it is found that

$$\Delta E_0=m^*\mathbf{v}_d{}^H\cdot(\mathbf{v}'-\mathbf{v}).$$

Substitution of this in (3.16) and further use of identity (3.12) results in

$$f_2=\left(-\frac{\partial^2 f_0}{\partial E^2}\right)\int_{-\infty}^{t}\left\{[m^*\mathbf{v}_d{}^H\cdot(\mathbf{v}'-\mathbf{v})]\left[\bar{e}\tau\left(\mathbf{\mathcal{E}}_s{}'-\frac{\mathbf{qqCu}'}{i\bar{e}\omega}+\frac{m^*\mathbf{u}'}{\bar{e}\tau}\right)\cdot\mathbf{v}'+\frac{2}{3}\frac{E_F{}^0}{N_0}N_1{}'\right]\right.$$
$$\left.-m^*v_d{}^H\bar{\omega}_c\tau\left[\bar{e}\tau\left(\mathbf{\mathcal{E}}_s{}'-\frac{\mathbf{qqCu}'}{i\bar{e}\omega}\right)\cdot\mathbf{v}'\right]\left[\frac{\bar{\omega}_c\tau v_x{}'+v_y{}'}{1+(\omega_c\tau)^2}\right]\right\}e^{-(t-t')/\tau}\frac{dt'}{\tau}. \quad (3.17)$$

The desired constitutive equation may now be found. Since $\mathbf{\mathcal{E}}_s{}'$, $\mathbf{N}_1{}'$, and $\mathbf{u}'$ all vary as $\exp[i(\mathbf{q}\cdot\mathbf{r}'-\omega t')]$ use of (3.3), (3.15), and (3.17) results in

$$\mathbf{j}_s=\boldsymbol{\sigma}\cdot\left[\mathbf{\mathcal{E}}_s-\frac{\mathbf{qqC}}{i\bar{e}\omega}\mathbf{u}+\frac{m^*\mathbf{u}}{\bar{e}\tau}\right]+\boldsymbol{\Sigma}\cdot\left[\mathbf{\mathcal{E}}_s-\frac{\mathbf{qqC}}{i\bar{e}\omega}\mathbf{u}\right]+\mathbf{R}N_1\bar{e}v_s, \quad (3.18)$$

where $\mathbf{j}_s$ is the ac part of the current of either electrons or holes and where

$$\boldsymbol{\sigma}_{ij}=\bar{e}^2\tau\int d^3v\int_{-\infty}^{t}\frac{dt'}{\tau}e^h v_i v_j{}'\left[\left(-\frac{\partial f_0}{\partial E}\right)+m^*\mathbf{v}_d{}^H\cdot(\mathbf{v}'-\mathbf{v})\left(-\frac{\partial^2 f_0}{\partial E^2}\right)\right],$$
$$\boldsymbol{\Sigma}_{ij}=-\frac{\bar{e}^2\tau v_d{}^H(\bar{\omega}_c\tau)}{1+(\bar{\omega}_c\tau)^2}\int d^3v\int_{-\infty}^{t}\frac{dt'}{\tau}e^h v_i\left[\left(-\frac{\partial f_0}{\partial E}\right)(\bar{\omega}_c\tau\delta_{xj}+\delta_{yj})+\left(-\frac{\partial^2 f_0}{\partial E^2}\right)m^* v_j{}'(\bar{\omega}_c\tau v_x{}'+v_y{}')\right], \quad (3.19)$$
$$\mathbf{R}=\frac{2}{3}\frac{E_F{}^0}{N_0 v_s}\int d^3v\int_{-\infty}^{t}\frac{dt'}{\tau}e^h\mathbf{v}\left[\left(-\frac{\partial f_0}{\partial E}\right)+\left(-\frac{\partial^2 f_0}{\partial E^2}\right)m^*\mathbf{v}_d{}^H\cdot(\mathbf{v}'-\mathbf{v})\right].$$

In these expressions the function $h(t')=(1-i\omega\tau)(t'-t)/\tau+i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})$. Use of the identities:

$$\int g(\mathbf{v})\left(-\frac{\partial f_0}{\partial E}\right)d^3v=\frac{3}{8\pi}\frac{N_0}{E_F{}^0}\int d\Omega\, g(\mathbf{v}_F),$$
$$\int g(\mathbf{v})\left(-\frac{\partial^2 f_0}{\partial E^2}\right)d^3v=-\frac{3}{8\pi}\frac{N_0}{E_F{}^0}\int d\Omega\frac{1}{m^* v_F{}^2}\frac{d}{dv_F}[v_F g(\mathbf{v}_F)], \quad (3.20)$$

gives the following results:

$$\boldsymbol{\sigma}_{ij}=\frac{3\sigma_0}{v_F{}^2}\int\frac{d\Omega}{4\pi}\int_{-\infty}^{t}\frac{dt'}{\tau}\left\{v_i v_j{}'e^h+\frac{1}{v^2}\frac{d}{dv}v[\mathbf{v}_d{}^H\cdot(\mathbf{v}-\mathbf{v}')]v_i v_j{}'e^h\right\}\Bigg|_{\mathbf{v}=\mathbf{v}_F}$$
$$\boldsymbol{\Sigma}_{ij}=-\frac{3\sigma_0}{v_F{}^2}\frac{v_d{}^H(\bar{\omega}_c\tau)}{1+(\bar{\omega}_c\tau)^2}\int\frac{d\Omega}{4\pi}\int_{-\infty}^{t}\frac{dt'}{\tau}\left\{v_i(\bar{\omega}_c\tau\delta_{xj}+\delta_{yj})e^h-\frac{1}{v^2}\frac{d}{dv}vv_i v_j{}'(\bar{\omega}_c\tau v_x{}'+v_y{}')e^h\right\}\Bigg|_{\mathbf{v}=\mathbf{v}_F}$$
$$\mathbf{R}=\frac{1}{v_s}\int\frac{d\Omega}{4\pi}\int_{-\infty}^{t}\frac{dt'}{\tau}\left\{\mathbf{v}e^h+\frac{1}{v^2}\frac{d}{dv}\bigg|v\mathbf{v}[\mathbf{v}_d{}^H\cdot(\mathbf{v}-\mathbf{v}')]e^h\right\}\Bigg|_{\mathbf{v}=\mathbf{v}_F}. \quad (3.21)$$

In these expressions $\sigma_0=N_0e^2\tau/m^*$.

The equation of continuity gives a relation between N_1 and the current, namely,

$$\bar{e}(\partial N/\partial t)+\mathrm{div}\mathbf{j}=0, \quad (3.22)$$

where N and $\mathbf{j}$ are the density and current of either electrons or holes. Thus

$$(\mathbf{j}_s\cdot\hat{q})=N_1\bar{e}v_s \quad (3.23)$$

and consequently

$$\mathbf{R}N_1\bar{e}v_s=\mathbf{R}(\hat{q}\cdot\mathbf{j}_s)=\mathbf{R}\cdot\mathbf{j}_s, \quad (3.24)$$

where $\mathbf{R}_{ij}=\mathbf{R}_i\hat{q}_j$.

The constitutive equation (3.18) will now be rewritten for both electrons and holes, using (3.24). In the following, e is the absolute value of the electronic charge.

$$\mathbf{j}_e=\boldsymbol{\sigma}_e'\cdot\left[\boldsymbol{\mathcal{E}}_s+\frac{\mathbf{qq}\mathbf{C}_e}{ie\omega}\mathbf{u}-\frac{m_e\mathbf{u}}{e\tau_e}\right]+\boldsymbol{\Sigma}_e'\cdot\left[\boldsymbol{\mathcal{E}}_s+\frac{\mathbf{qq}\mathbf{C}_e}{ie\omega}\mathbf{u}\right]$$
$$\mathbf{j}_h=\boldsymbol{\sigma}_h'\cdot\left[\boldsymbol{\mathcal{E}}_s-\frac{\mathbf{qq}\mathbf{C}_h}{ie\omega}\mathbf{u}+\frac{m_h\mathbf{u}}{e\tau_h}\right]+\boldsymbol{\Sigma}_h'\cdot\left[\boldsymbol{\mathcal{E}}_s-\frac{\mathbf{qq}\mathbf{C}_h}{ie\omega}\mathbf{u}\right]. \quad (3.25)$$

In these equations, $\mathbf{j}_e$ and $\mathbf{j}_s$ are the ac parts of the currents, and

$$\boldsymbol{\sigma}_e'=(\mathbf{I}-\mathbf{R}_e)^{-1}\boldsymbol{\sigma}_e \quad \boldsymbol{\Sigma}_e'=(\mathbf{I}-\mathbf{R}_e)^{-1}\boldsymbol{\Sigma}_e$$
$$\boldsymbol{\sigma}_h'=(\mathbf{I}-\mathbf{R}_h)^{-1}\boldsymbol{\sigma}_h \quad \boldsymbol{\Sigma}_h'=(\mathbf{I}-\mathbf{R}_h)^{-1}\boldsymbol{\Sigma}_h.$$

B. The Attenuation (Amplification) Coefficient

The net power dissipated per unit volume from the sound wave is, as shown in Ref. 14:

$$Q=\tfrac{1}{2}\,\mathrm{Re}\left\{\mathbf{j}_e^*\cdot\left[\boldsymbol{\mathcal{E}}_s+\frac{\mathbf{qq}\mathbf{C}_e}{ie\omega}\mathbf{u}\right]+\mathbf{j}_h^*\cdot\left[\boldsymbol{\mathcal{E}}_s-\frac{\mathbf{qq}\mathbf{C}_h}{ie\omega}\mathbf{u}\right]-(Nm/\tau_e)[(\langle\mathbf{v}_e\rangle-\mathbf{u})^*\cdot\mathbf{u}]-(Nm/\tau_h)[(\langle\mathbf{v}_h\rangle-\mathbf{u})^*\cdot\mathbf{u}]\right\}$$
$$=\tfrac{1}{2}\,\mathrm{Re}\,\mathbf{u}^*\left[-(\mathbf{C}_e-\mathbf{C}_h)^\dagger\frac{\mathbf{qq}}{2ie\omega}(\mathbf{j}_e+\mathbf{j}_h)-(\mathbf{C}_e+\mathbf{C}_h)^\dagger\frac{\mathbf{qq}}{2ei\omega}(\mathbf{j}_e-\mathbf{j}_h)+\frac{m}{e\tau_e}\mathbf{j}_e-\frac{m}{e\tau_h}\mathbf{j}_h+Nm\left(\frac{1}{\tau_e}+\frac{1}{\tau_h}\right)\mathbf{u}\right]. \quad (3.26)$$

In these expressions m is the actual electron mass (and not an effective mass) as discussed in Ref. 14.

The first two terms in the right-hand side of (3.26) which are proportional to the deformation potential tensor are of order of magnitude $Cq^2\tau/m\omega$ relative to the third and fourth terms. For a reasonable value of the deformation potential, say, $C\approx 10$ eV

$$\frac{Cq^2\tau}{m\omega}=\frac{C}{mv_s^2}\omega\tau\approx 5\times 10^5\omega\tau. \quad (3.27)$$

Therefore, except for the very lowest frequencies (in the Kc range) the third and fourth terms may be neglected, and this will be done in subsequent calculations.

Use of Maxwell's equations (3.1) and the constitutive equations (3.25) gives expressions for the currents in terms of $\mathbf{u}$.

$$\mathbf{j}_e+\mathbf{j}_h=\mathbf{B}(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}\left[(\mathbf{S}_e'-\mathbf{S}_h')\frac{\mathbf{qq}}{2ei\omega}(\mathbf{C}_e+\mathbf{C}_h)+(\mathbf{S}_e'+\mathbf{S}_h')\frac{\mathbf{qq}}{2ie\omega}(\mathbf{C}_e-\mathbf{C}_h)-\frac{m_e}{e\tau_e}\boldsymbol{\sigma}_e'+\frac{m_h}{e\tau_h}\boldsymbol{\sigma}_h'\right]\mathbf{u}, \quad (3.28)$$

$$\mathbf{j}_e-\mathbf{j}_h=\left\{[(\mathbf{S}_e'+\mathbf{S}_h')-(\mathbf{S}_e'-\mathbf{S}_h')(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}(\mathbf{S}_e'-\mathbf{S}_h')]\frac{\mathbf{qq}}{2ei\omega}(\mathbf{C}_e+\mathbf{C}_h)\right.$$
$$+(\mathbf{S}_e'-\mathbf{S}_h')(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}\mathbf{B}\frac{\mathbf{qq}}{2ie\omega}(\mathbf{C}_e-\mathbf{C}_h)-\frac{m_e}{e\tau_e}[\mathbf{I}-(\mathbf{S}_e'-\mathbf{S}_h')(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}\boldsymbol{\sigma}_e']$$
$$\left.-\frac{m_h}{e\tau_h}[\mathbf{I}+(\mathbf{S}_e'-\mathbf{S}_h')(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}\boldsymbol{\sigma}_h']\right\}\mathbf{u}. \quad (3.29)$$

In these expressions the tensors $\mathbf{S}_{e,h}'$ are defined by

$$\mathbf{S}_{e,h}=\boldsymbol{\sigma}_{e,h}+\boldsymbol{\Sigma}_{e,h},$$
$$\mathbf{S}_{e,h}'=\boldsymbol{\sigma}_{e,h}'+\boldsymbol{\Sigma}_{e,h}'. \quad (3.30)$$

Since the tensors $\boldsymbol{\sigma}$ are of the same order of magnitude as the tensors $\mathbf{S}$ (this will later be verified by direct computation of the tensors) all terms in $\mathbf{j}_e\pm\mathbf{j}_h$ may be neglected, except for those proportional to the deformation potential tensors, because $Cq^2\tau/m\omega\gg 1$, as discussed above. Also, the second term in Eq. (3.29) for $\mathbf{j}_e-\mathbf{j}_h$ contains as a factor the tensor $\mathbf{Bqq}$. It is easy to show that $\mathbf{Bqq}=\mathbf{qqB}=-i\gamma\mathbf{qq}$. This term will be neglected because it is of order of magnitude $(v_s/c)^2$ relative to β. Similarly, when (3.28) is substituted in the formula for net power dissipated (3.26), the resulting term will be proportional to $\mathbf{qqB}$ and is therefore negligible. After these approximations are made, the formula for net power dissipated becomes

$$Q=\frac{Nm}{2}\left(\frac{1}{\tau_e}+\frac{1}{\tau_h}\right)|\mathbf{u}|^2+\frac{1}{8e^2\omega^2}\operatorname{Re}\mathbf{u}^*(\mathbf{C}_e+\mathbf{C}_h)^\dagger$$
$$\times\mathbf{qq}[(\mathbf{S}_e'+\mathbf{S}_h')-(\mathbf{S}_e'-\mathbf{S}_h')(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}(\mathbf{S}_e'-\mathbf{S}_h')]\mathbf{qq}(\mathbf{C}_e+\mathbf{C}_h)\mathbf{u}. \quad (3.31)$$

The coefficient of attenuation α is found by using

$$\alpha=Q/(\tfrac{1}{2}\rho|\mathbf{u}|^2v_s), \quad (3.32)$$

where ρ is the mass density of the semimetal. If $\mathbf{u}$ is polarized along the jth axis, and the 1 axis is in the direction $\hat{q}$, then

$$\alpha_j=\frac{q^4(\mathbf{C}_e+\mathbf{C}_h)_{1j}^2}{4e^2\omega^2\rho v_s}\operatorname{Re}[(\mathbf{S}_e'+\mathbf{S}_h')-(\mathbf{S}_e'-\mathbf{S}_h')(\mathbf{S}_e'+\mathbf{S}_h'+\mathbf{B})^{-1}(\mathbf{S}_e'-\mathbf{S}_h')]_{11}+\frac{Nm}{\rho v_s}\left(\frac{1}{\tau_e}+\frac{1}{\tau_h}\right), \quad (3.33)$$

where α_j is the coefficient of attenuation for sound polarized in the jth direction.

C. The Conductivity Tensor

The coefficient of attenuation is now completely specified in terms of the conductivity tensors and the vectors $\mathbf{R}_e$ and $\mathbf{R}_h$, and it now remains to calculate these quantities explicitly. Inspection of Eq. (3.21) shows that $\boldsymbol{\sigma}$, $\boldsymbol{\Sigma}$, and $\mathbf{R}$ are linear combinations of terms of the form

$$\int\frac{d\Omega}{4\pi}\int_{-\infty}^{t}\frac{dt'}{\tau}e^hG(\mathbf{v})H(\mathbf{v}')=\bar{G}\bar{H}, \quad (3.34)$$

where $G(\mathbf{v})$ and $H(\mathbf{v}')$ are products of their arguments. Solution of the equations of motion for electrons and holes in crossed electric and magnetic fields results in

$$\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})=\mathbf{q}\cdot\mathbf{v}_c(t'-t)+\mathbf{X}\cdot(\mathbf{v}'-\mathbf{v}), \quad (3.35)$$

where $\mathbf{v}_c$ is the sum of the drift velocities

$$\mathbf{v}_c=\mathbf{v}_d{}^H+v_z\hat{H},$$

and

$$\mathbf{X}=\mathbf{q}\times\mathbf{H}/\bar{\omega}_c,$$

and $\hat{H}$ is a unit vector in the direction of the magnetic field. Therefore,

$$h(t')=(1-i\omega\tau)(t'-t)/\tau+i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})=(1-i\omega\tau+i\mathbf{q}\cdot\mathbf{v}_c\tau)(t'-t)/\tau+i\mathbf{X}\cdot(\mathbf{v}'-\mathbf{v})=\Lambda s/\tau+i\mathbf{X}\cdot(\mathbf{v}'-\mathbf{v}), \quad (3.36)$$

where $\Lambda=1-i\omega\tau+i\mathbf{q}\cdot\mathbf{v}_c\tau$ and $s=t'-t$. Substitution of (3.36) in (3.34) gives

$$\bar{G}\bar{H}=\int\frac{d\Omega}{4\pi}\int_{-\infty}^{0}\frac{ds}{\tau}[G(\mathbf{v})\exp(i\mathbf{X}\cdot\mathbf{v})]^*[H(\mathbf{v}')\exp(i\mathbf{X}\cdot\mathbf{v}')]e^{\Lambda s/\tau}$$
$$=\int\frac{d\Omega}{4\pi}\int_{-\infty}^{0}\frac{ds}{\tau}[G(-i\boldsymbol{\nabla}_\mathbf{x},v_z)\exp(i\mathbf{X}\cdot\mathbf{v})]^*[H(-i\boldsymbol{\nabla}_\mathbf{x},v_z)\exp(i\mathbf{X}\cdot\mathbf{v}')]e^{\Lambda s/\tau}. \quad (3.37)$$

The operator $\boldsymbol{\nabla}_\mathbf{x}$ has meaning only for the two directions perpendicular to $\mathbf{H}$, since $\mathbf{X}$ is always perpendicular to $\mathbf{H}$. The notation on the right hand side of (3.37) means that the vector $\mathbf{v}=(\mathbf{v}_\perp,v_z)$ is replaced by the vector $(-i\boldsymbol{\nabla}_\mathbf{x},v_z)$ in the functions G and H.

The equations of motion are now solved to find the vector $\mathbf{v}'$ in terms of $\mathbf{v}$ and s. First, define the angles ϕ', θ' and ϕ, θ as follows:

$$\mathbf{v}=v\begin{bmatrix}\sin\theta'\cos\phi'\\ \sin\theta'\sin\phi'\\ \cos\theta'\end{bmatrix},\quad \mathbf{q}=q\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}. \quad (3.38)$$

Then solution of the equations of motion of charge carriers in crossed electric and magnetic fields results in

$$\mathbf{X}\cdot\mathbf{v}'=Xv_d{}^H[\sin\phi-\sin(\bar{\omega}_cs+\phi)]+Xv\sin\theta'\sin(\bar{\omega}_cs-\phi'+\phi),$$
$$\mathbf{X}\cdot\mathbf{v}=Xv\sin\theta'\sin(\phi-\phi'). \quad (3.39)$$

The following identity is used to expand $\exp(i\mathbf{X}\cdot\mathbf{v})$ and $\exp(i\mathbf{X}\cdot\mathbf{v}')$.

$$\exp(iz\sin\psi)=\sum_{n=-\infty}^{\infty}J_n(x)\exp(iz\psi). \tag{3.40}$$

Thus,

$$\begin{aligned}\exp(i\mathbf{X}\cdot\mathbf{v})&=\sum_{n=-\infty}^{\infty}J_n(Xv\sin\theta')\exp[in(\phi-\phi')],\\ \exp(i\mathbf{X}\cdot\mathbf{v}')&=\{1+iXv_d{}^H[\sin\phi-\sin(\bar{\omega}_cs+\phi)]\}\sum_{n=-\infty}^{\infty}J_n(Xv\sin\theta')\exp[in(\bar{\omega}_cs-\phi'+\phi)].\end{aligned} \tag{3.41}$$

The last expression has been expanded to first order in $\mathbf{v}_d{}^H$ which corresponds to expansion to first order in $\mathbf{E}$. Substitution of (3.41) in (3.37) and integration over the angle ϕ' results in

$$\bar{G}\bar{H}=\sum_{n=-\infty}^{\infty}\int\frac{d(\cos\theta')}{2}\int_{-\infty}^{0}\frac{ds}{\tau}\exp[(\Lambda+in\bar{\omega}_c\tau)s/\tau][G(-i\nabla_{\mathbf{x}},v_z)J_n(Xv\sin\theta')\,e^{in\phi}]^*$$
$$\times[H(-i\nabla_{\mathbf{x}},v_z)J_n(Xv\sin\theta')\,e^{in\phi}\{1+iXv_d{}^H[\sin\phi+\sin(\bar{\omega}_cs+\phi)]\}]. \tag{3.42}$$

Integration over the variable s now gives:

$$\bar{G}\bar{H}=\sum_{n=-\infty}^{\infty}\int\frac{d(\cos\theta')}{2\Lambda_n}[G(-i\nabla_{\mathbf{x}},v_z)J_n(Xv\sin\theta')\,e^{in\phi}]^*$$
$$\times\left[H(-i\nabla_{\mathbf{x}},v_z)J_n(Xv\sin\theta')e^{in\phi}\left\{1+\frac{iv_d{}^H\bar{\omega}_c\tau}{\Lambda_n{}^2+(\bar{\omega}_c\tau)^2}[X_x\bar{\omega}_c\tau-\Lambda_nX_y]\right\}\right], \tag{3.43}$$

where $\Lambda_n=\Lambda+in\bar{\omega}_c\tau$. This final integration over the angle θ' may be performed by using different approximations depending upon the relative values of the parameters involved. A comparison of these expressions with Eq. (3.9) of Ref. 12 shows that the oscillatory phenomena of magnetoattenuation should also occur in the presence of an external electric field. Geometric resonances will occur for the same conditions as for zero electric field; cyclotron resonance will occur for $\omega-\mathbf{q}\cdot\mathbf{v}_d{}^H=n\omega_c$ (when $q\perp H$). It is expected from the arguments given in the introduction that amplification will occur for $\omega_c\tau v_d{}^H/[1+(\omega_c\tau)^2]^{1/2}\geqslant v_s$; and, therefore, the oscillatory phenomena will appear in amplification. These phenomena will not be discussed here, for we shall limit ourselves to a discussion of high magnetic field effects, i.e., $\omega_c\tau\gg1$.

The resonance behavior discussed in the introduction may be seen explicitly in (3.43). The denominator $\Lambda_n=iq\tau[(\mathbf{v}_c\cdot\hat{q}+n\omega_c/q-v_s)-i/q\tau]$ is actually the resonance denominator which is expected from energy-momentum conservation arguments. The extra term $n\omega_c/q$ is expected when electrons may jump to a different cyclotron orbit; but for the condition of high magnetic field discussed in the introduction ω_c/ω is too large to allow this jump. Therefore, only terms with $n=0$ should contribute to the resonant amplification or attenuation. Formally the condition of high magnetic field is expressed by expansion of $1/\Lambda_n$ in powers of $1/\omega_c\tau$

$$\frac{1}{\Lambda_n}=\frac{1}{\Lambda+in\omega_c\tau}=\frac{1}{in\omega_c\tau}\sum_{\nu=0}^{\infty}\left(\frac{i\Lambda}{n\omega_c\tau}\right)^{\nu},\qquad n\neq0$$
$$=\frac{1/iq\tau}{\{(\mathbf{v}_c\cdot\hat{q}-v_s)-i/q\tau\}},\qquad n=0.$$

Thus, contributions to the resonances come only from those terms with $n=0$.

It should be noted here that the resonance condition is of course not necessarily the condition for *peak* amplification or attenuation. Consider the simple resonant function $f=i\Gamma/[(E-E_0)+i\Gamma]$. Re$f$ peaks at $E=E_0$, but Imf peaks at $E=E_0\pm\Gamma$ (and, in fact, Im$f=0$ at $E=E_0$). Since $\mathrm{Im}(1/\Lambda_n)$ will in general, contribute to the amplification, the peak amplification is expected to be shifted somewhat from the position where the resonance condition is fulfilled.

For high magnetic field, the conductivity tensors were calculated in the coordinate system in which the z axis is in the direction of $\mathbf{H}$, and the y axis in the direction of $\mathbf{E}$, by using (3.21) and (3.43). The coordinate system was then rotated so that the 1 axis was in the direction $\mathbf{q}$, the 2 axis in the direction $\mathbf{H}\times\mathbf{q}$, and the 3 axis in the direction $\mathbf{q}\times[\mathbf{H}\times\mathbf{q}]$. In this coordinate system the conductivity tensors become (to the second order in $1/\omega_c\tau$):

$$
\begin{aligned}
\frac{\sigma_{11}}{\sigma_0}&=\frac{A}{\lambda^2}\cos^2\theta(1-i\omega\tau)-\frac{1}{\bar{\omega}_c\tau}\frac{iq_yv_d\tau}{\lambda}A\cos^2\theta+\frac{1}{(\bar{\omega}_c\tau)^2}(\lambda-\tfrac{1}{2}iq_xv_d\tau)\sin^2\theta(1-B\cos^2\theta),\\
\frac{\sigma_{12}}{\sigma_0}&=\frac{1}{\bar{\omega}_c\tau}\sin\theta\frac{(1-i\omega\tau)}{\lambda}(1-B\cos^2\theta)-\frac{1}{(\bar{\omega}_c\tau)^2}\tfrac{3}{2}iq_yv_d\tau\sin\theta(1-B\cos^2\theta),\\
\frac{\sigma_{13}}{\sigma_0}&=\frac{A}{\lambda^2}\sin\theta\cos\theta(1-i\omega\tau)-\frac{1}{\bar{\omega}_c\tau}\frac{iq_yv_d\tau}{\lambda}A\sin\theta\cos\theta-\frac{1}{(\bar{\omega}_c\tau)^2}\sin\theta\cos\theta[(\lambda-\tfrac{1}{2}iq_xv_d\tau)(1+B\sin^2\theta)-\tfrac{1}{2}iBq_xv_d\tau],\\
\frac{\sigma_{21}}{\sigma_0}&=\frac{A}{\lambda^2}iqv_d\tau\cos^2\theta\sin\phi-\frac{1}{\bar{\omega}_c\tau}\sin\theta\left(1+\frac{iq_xv_d\tau}{\lambda}\right)(1-B\cos^2\theta)+\frac{1}{(\bar{\omega}_c\tau)^2}[iqv_d\tau\sin\phi-iq_yv_d\tau\sin\theta(\tfrac{5}{2}+3B\cos^2\theta)],\\
\frac{\sigma_{22}}{\sigma_0}&=\frac{1}{\bar{\omega}_c\tau}\frac{2iq_yv_d\tau}{\lambda}(1-B\cos^2\theta)+\frac{1}{(\bar{\omega}_c\tau)^2}\left[\lambda-\tfrac{1}{2}iq_xv_d\tau+\frac{(ql)^2}{2\lambda}\sin^2\theta(1-B\cos^2\theta)-\frac{\lambda}{2}B\sin^2\theta\right],\\
\frac{\sigma_{23}}{\sigma_0}&=\frac{A}{\lambda^2}iq_yv_d\tau\cos\theta+\frac{1}{\bar{\omega}_c\tau}\cos\theta\left[B+\left(1+\frac{iq_xv_d\tau}{\lambda}\right)(1-B\cos^2\theta)\right]+\frac{1}{(\bar{\omega}_c\tau)^2}iq_yv_d\tau\cos\theta(\tfrac{5}{2}-3B\sin^2\theta),\\
\frac{\sigma_{31}}{\sigma_0}&=\frac{A}{\lambda^2}\cos\theta[(1-i\omega\tau)\sin\theta+iqv_d\tau\cos\phi]-\frac{1}{\bar{\omega}_c\tau}\frac{iq_yv_d\tau}{\lambda}A\sin\theta\cos\theta\\
&\qquad-\frac{1}{(\bar{\omega}_c\tau)^2}\cos\theta[\sin\theta(\lambda-\tfrac{1}{2}iq_xv_d\tau)(1+B\sin^2\theta)+iqv_d\tau\cos\phi(1-B\sin^2\theta)],\\
\frac{\sigma_{32}}{\sigma_0}&=-\frac{1}{\bar{\omega}_c\tau}\cos\theta\left[B+\frac{(1-i\omega\tau)}{\lambda}(1-B\cos^2\theta)\right]+\frac{1}{(\bar{\omega}_c\tau)^2}\tfrac{3}{2}iq_yv_d\tau\cos\theta(1+B\sin^2\theta),\\
\frac{\sigma_{33}}{\sigma_0}&=\frac{A}{\lambda}\left[1-\frac{\cos^2\theta}{\lambda}(1-i\omega\tau)\right]-\frac{1}{\bar{\omega}_c\tau}\frac{iq_yv_d\tau}{\lambda}A\sin^2\theta+\frac{1}{(\bar{\omega}_c\tau)^2}[(\lambda-\tfrac{1}{2}iq_xv_d\tau)\cos^2\theta(1+B\sin^2\theta)+(iq_xv_d\tau-B\lambda\sin^2\theta)].
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
\frac{S_{11}}{\sigma_0}&=\frac{A}{\lambda^2}\cos^2\theta(1-i\omega\tau)+\frac{1}{(\bar{\omega}_c\tau)^2}\left[(1-i\omega\tau)(1-B\cos^2\theta)\sin^2\theta-\frac{iq_xv_d\tau}{\lambda}A\cos^2\theta\right],\\
\frac{S_{12}}{\sigma_0}&=\frac{1}{\bar{\omega}_c\tau}\sin\theta\frac{(1-i\omega\tau)}{\lambda}(1-B\cos^2\theta),\\
\frac{S_{13}}{\sigma_0}&=\frac{A}{\lambda^2}\sin\theta\cos\theta(1-i\omega\tau)-\frac{1}{(\bar{\omega}_c\tau)^2}\sin\theta\cos\theta\left[\frac{A}{\lambda}iq_xv_d\tau+(1-i\omega\tau)(1+B\sin^2\theta)\right],\\
\frac{S_{21}}{\sigma_0}&=\frac{A}{\lambda^2}iqv_d\tau\cos^2\theta\sin\phi-\frac{1}{\bar{\omega}_c\tau}\sin\theta(1-B\cos^2\theta)+\frac{1}{(\bar{\omega}_c\tau)^2}iq_yv_d\tau\sin\theta(1-B\cos^2\theta)\left(1+\frac{1}{\lambda}\right),\\
\frac{S_{22}}{\sigma_0}&=\frac{1}{\bar{\omega}_c\tau}\frac{iq_yv_d\tau}{\lambda}(1-B\cos^2\theta)+\frac{1}{(\bar{\omega}_c\tau)^2}\left\{\lambda(1-\tfrac{1}{2}B\sin^2\theta)+\left(\frac{iq_xv_d\tau}{\lambda}+\frac{(ql)^2}{2\lambda}\sin^2\theta\right)(1-B\cos^2\theta)\right\},\\
\frac{S_{23}}{\sigma_0}&=\frac{A}{\lambda^2}iq_yv_d\tau\cos\theta+\frac{1}{\bar{\omega}_c\tau}\cos\theta(1+B\sin^2\theta)-\frac{1}{(\bar{\omega}_c\tau)^2}iq_yv_d\tau\cos\theta\left[(1-B\cos^2\theta)\left(1+\frac{1}{\lambda}\right)+B\right],\\
\frac{S_{31}}{\sigma_0}&=\frac{A}{\lambda^2}[\sin\theta\cos\theta(1-i\omega\tau)+iqv_d\tau\cos\theta\cos\phi]-\frac{1}{(\bar{\omega}_c\tau)^2}\sin\theta\cos\theta\left[(1-i\omega\tau)(1-B\cos^2\theta)+B\lambda+\frac{A}{\lambda}iq_xv_d\tau\right],
\end{aligned}
\tag{3.45}
$$

$$\frac{S_{32}}{\sigma_0}=-\frac{1}{\bar{\omega}_c\tau}\cos\theta\left[\frac{(1-i\omega\tau)}{\lambda}(1-B\cos^2\theta)+B\right],$$

$$\frac{S_{33}}{\sigma_0}=\frac{A}{\lambda}\left[1-\cos^2\theta\frac{(1-i\omega\tau)}{\lambda}\right]+\frac{1}{(\bar{\omega}_c\tau)^2}\left[(1-i\omega\tau)\cos^2\theta(1+B\sin^2\theta)-\left(\frac{A}{\lambda}iq_xv_d\tau+B\lambda\right)\sin^2\theta\right].$$

$$R_1=\frac{1}{\omega\tau}\left\{\frac{q_xv_d\tau}{\lambda}-\frac{i(ql)^2}{3\lambda^3}A\cos^2\theta(1-i\omega\tau)+\frac{1}{\bar{\omega}_c\tau}q_yv_d\tau\left(1-\frac{(ql)^2}{3\lambda^2}A\cos^2\theta\right)\right. \tag{3.46}$$

$$\left.-\frac{1}{(\bar{\omega}_c\tau)^2}\left[\lambda q_xv_d\tau+\frac{i(ql)^2}{3\lambda}\sin^2\theta(1-i\omega\tau)(1-B\cos^2\theta)\right]\right\},$$

$$R_2=\frac{1}{\omega\tau}\left\{-\frac{qv_d\tau}{\lambda}\sin\phi\left(1-\frac{(ql)^2}{3\lambda^2}A\cos^2\theta\right)+\frac{1}{\bar{\omega}_c\tau}\left[qv_d\tau\cos\phi+\frac{i(ql)^2}{3\lambda}\sin\theta(1-B\cos^2\theta)\right]\right.$$

$$\left.+\frac{1}{(\bar{\omega}_c\tau)^2}\left[\lambda qv_d\tau\sin\phi+\frac{(ql)^2}{\lambda}q_yv_d\tau\sin\theta(1-B\cos^2\theta)\right]\right\},$$

$$R_3=\frac{1}{\omega\tau}\left\{-\frac{qv_d\tau}{\lambda}\cos\theta\cos\phi\left[1-\frac{(ql)^2}{3\lambda^2}A\right]-\frac{i(ql)^2}{3\lambda^3}A\cos\theta\sin\theta(1-i\omega\tau)-\frac{1}{\bar{\omega}_c\tau}qv_d\tau\cos\theta\sin\phi\left[1+\frac{(ql)^2}{3\lambda^2}A\sin^2\theta\right]\right.$$

$$\left.+\frac{1}{(\bar{\omega}_c\tau)^2}\left[\lambda qv_d\tau\cos\theta\cos\phi+\frac{i(ql)^2}{3}\sin\theta\cos\theta\left\{B+\frac{(1-i\omega\tau)}{\lambda}(1-B\cos^2\theta)\right\}\right]\right\},$$

where

$$A=3(1-I)/\alpha^2;\quad B=[(ql)^2/2\lambda][A(1+1/\alpha^2)-1/\alpha^2];\quad \lambda=1-i\omega\tau+i\mathbf{q}\cdot\mathbf{v}_d\tau;$$

$$\alpha=ql\cos\theta/\lambda \quad\text{and}\quad I=\int_{-1}^{1}dx/2(1+i\alpha x).$$

We note that $A\to 1$ as $\alpha\to 0$, and $B\to (ql)^2/5\lambda^2$ as $\alpha\to 0$. In the above expressions $\mathbf{v}_d=\mathbf{v}_d^H$. We will continue to drop the superscript in the next section.

IV. RESULTS AND DISCUSSION

We specialize to the case $m_e=m_h=m^*$, and $\tau_e=\tau_h=\tau$; and calculate the attenuation to lowest order in $1/\omega_c\tau$. First consider the case where $\mathbf{q}$ is not perpendicular to $\mathbf{H}$, i.e., $\cos\theta\neq 0$. In this case $(S_e'+S_h')_{11}$ has components to zeroth order in $1/\omega_c\tau$; but $(S_e'-S_h')$ is of order $1/\omega_c\tau$, so that the second term in the coefficient of attenuation (3.33) is of order $1/(\omega_c\tau)^2$ relative to the first. Dropping this term [and also terms of order $1/(\omega_c\tau)^2$ in $(S_e'+S_h')_{11}$] corresponds to neglecting $1/(\omega_c\tau)^2$ in comparison with $\cos^2\theta$. For $\omega_c\tau=1000$, which is easily attainable experimentally, $\cos^2\theta\gg 1/(\omega_c\tau)^2$ for all θ except $89.5°\lesssim\theta\lesssim 90.5°$.

Using the above approximations, $\mathbf{S}_e=\mathbf{S}_h$, $\mathbf{R}_e=\mathbf{R}_h$, and

$$(S_e'+S_h')_{11}=\frac{2S_{11}}{1-R_1}=\frac{2\sigma_0\omega\tau}{\lambda}\frac{A\cos^2\theta}{(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)+[i(ql)^2/3\lambda^2]A\cos^2\theta}$$
$$=\frac{6\sigma_0\omega\tau}{(ql)^2}(-i\lambda)\left[1+\frac{i(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)}{\lambda-I}\right] \tag{4.1}$$

and

$$\mathrm{Re}(S_e'+S_h')_{11}=\frac{6\sigma_0\omega\tau}{(ql)^2}(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)\left[\mathrm{Re}\frac{1}{1-(I/\lambda)}-1\right]$$
$$=\frac{6\sigma_0\omega\tau}{(ql)^2}(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)\left[\frac{R}{R^2+S^2}-1\right], \tag{4.2}$$

where $1-I/\lambda=R-iS$ and

$$R=1-\frac{1}{2ql\cos\theta}\{\arctan[ql\cos\theta+(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)]+\arctan[ql\cos\theta-(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)]\}, \tag{4.3}$$

$$S=\frac{1}{4ql\cos\theta}\ln\frac{1+[ql\cos\theta+(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)]^2}{1+[ql\cos\theta-(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)]^2}.$$

The coefficient of attenuation is found by using (3.33)

$$\alpha_j=\frac{2Nm}{\rho v_s\tau}[1+aF(\mathbf{q},\mathbf{v}_d)], \tag{4.4}$$

where

$$a=\frac{3mv_F}{m^*v_s}\left[\frac{(\mathbf{C}_e+\mathbf{C}_h)_{1j}}{2mv_F^2}\right]^2$$

and

$$F(\mathbf{q},\mathbf{v}_d)=ql(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)\left(\frac{R}{R^2+S^2}-1\right).$$

Using typical values of the parameters for bismuth

$$m/m^* \approx 10\,, \quad v_F/v_s \approx 50\,, \quad [(C_e+C_h)/2mv_F^2] \approx 20\,,$$ [14]

the value of a is found to be $\approx 6\times 10^5$.

The functions R and S^2 are even in $(\omega\tau - \mathbf{q}\cdot\mathbf{v}_d\tau)$ so $F(\mathbf{q},\mathbf{v}_d)$ is negative for $\hat{q}\cdot\mathbf{v}_d \gtrsim v_s$. However, because of the first term in α_j, amplification begins not at $\hat{q}\cdot\mathbf{v}_d = v_s$, but at a slightly higher value of $\hat{q}\cdot\mathbf{v}_d$.

Figure 3 is a sketch of the geometry of the system. The z axis is in the direction of $\mathbf{H}$, the y axis in the direction $\mathbf{E}$, and the x axis in the direction $\mathbf{v}_d$. The cone is defined by $\hat{q}\cdot\mathbf{v}_d = v_s$. The function $F(\mathbf{q},\mathbf{v}_d)$ is negative for all directions of propagation $\hat{q}$ within this cone. The angle θ is the angle between $\hat{q}$ and the z axis; ϕ is the angle between the xz plane and the plane determined by the z axis and $\mathbf{q}$. We note that $F(\mathbf{q},\mathbf{v}_d)$ is a function of $\mathbf{v}_d$ only through the combination $v_d\cos\phi$, and therefore define $v_{d,\mathrm{eff}} = v_d\cos\phi$. For $\phi=0$ (the xz plane) $F(\mathbf{q},\mathbf{v}_d)$ is negative when $\theta_0<\theta<\pi-\theta_0$, where $\sin\theta_0 = v_s/v_d$; for a plane at angle ϕ, $F(\mathbf{q},\mathbf{v}_d)$ is negative when $\theta'<\theta<\pi-\theta'$, where $\sin\theta' = v_s/v_{d,\mathrm{eff}}$.

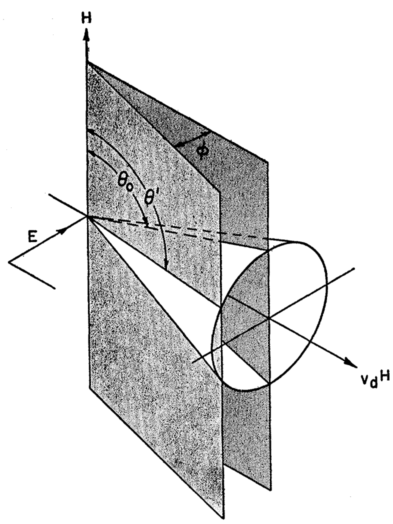

FIG. 3. Geometry for crossed field case. Amplification occurs for all directions of propagation $\hat{q}$ within the cone, which is defined by $\hat{q}\cdot\mathbf{v}_d^H = v_s$.

Figure 4 is a plot of $F(\mathbf{q},\mathbf{v}_d)/ql$ as a function of the angle θ in the plane at angle ϕ to the xz plane, for several values of $v_{d,\mathrm{eff}}$, with the parameter values $\omega\tau=0.5$, $v_F/v_s=50$. Inspection of these curves shows that $aF(\mathbf{q},\mathbf{v}_d)\gg 1$, except for a very small region close to $\hat{q}\cdot\mathbf{v}_d=v_s$. Therefore, over most of the range, the attenuation (amplification) is proportional to $F(\mathbf{q},\mathbf{v}_d)$. The sharp resonance peaks at values of θ close to 90° are, of course, the resonances corresponding to $\hat{q}\cdot\mathbf{v}_d \pm v_F(\hat{q}\cdot\hat{H}) = v_s$. As $v_{d,\mathrm{eff}}/v_s$ decreases, the peaks in amplification, corresponding to $\hat{q}\cdot\mathbf{v}_d - v_F(\hat{q}\cdot\hat{H}) \approx v_s$, sharpen and move closer to the perpendicular to the magnetic field. There are no resonances for $v_{d,\mathrm{eff}}/v_s=1$; for $v_{d,\mathrm{eff}}/v_s<1$ attenuation peaks appear, corresponding to $\hat{q}\cdot\mathbf{v}_d + v_F(\hat{q}\cdot\hat{H}) \approx v_s$. The width of the peaks is a rather sensitive function of ql; the peaks tend to sharpen as ql increases, as would be expected from the qualitative theory. Spector[11] examined the special case $v_d=0$, and found that for large ql the inflection point between the resonance peak and $\theta=\frac{1}{2}\pi$ corresponds to $v_F(\hat{q}\cdot\hat{H}) = v_s$. A similar situation should exist in the more general case.

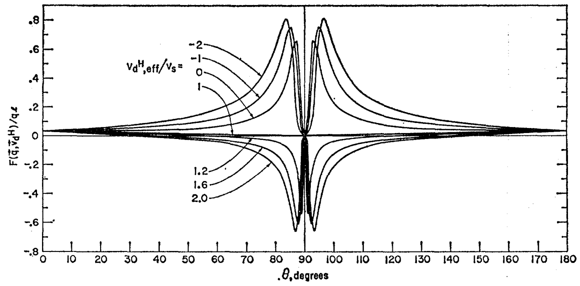

FIG. 4. Crossed electric and magnetic fields: $F(\mathbf{q},\mathbf{v}_d^H)/ql$ as a function of the angle between the direction of propagation of sound and the magnetic field, for several values of $v_{d,\mathrm{eff}}^H/v_s$. The parameter values used are $ql=25$, $v_F/v_s=50$. The attenuation is proportional to $[1+aF(\mathbf{q},\mathbf{v}_d^H)]$.

In the limiting case $ql\cos\theta \ll 1$, $\omega\tau - \mathbf{q}\cdot\mathbf{v}_d\tau \ll 1$

$$F(\mathbf{q},\mathbf{v}_d) = ql(\omega\tau - \mathbf{q}\cdot\mathbf{v}_d\tau) \times\left[\frac{\frac{1}{3}(ql\cos\theta)^2}{(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)^2 + [\frac{1}{3}(ql\cos\theta)^2]^2}\right]. \quad (4.5)$$

This limiting case corresponds to $v_d/v_s \to 1$, $\theta\to\frac{1}{2}\pi$ (but $\cos\theta \gg 1/\omega_c\tau$); or to $ql\ll 1$. For these cases $F(\mathbf{q},\mathbf{v}_d)$ has peak values of $\pm\frac{1}{2}ql$ at $|\omega\tau - \mathbf{q}\cdot\mathbf{v}_d\tau| = \frac{1}{3}(ql\cos\theta_0)^2 \approx 0$, where $\sin\theta_0 = v_s/v_{d,\mathrm{eff}}$. Figure 5 is a plot of $F(\mathbf{q},\mathbf{v}_d)/ql$ for $ql=0.01$ and several values of $v_{d,\mathrm{eff}}/v_s$.

Figure 6 is a plot of the peak value of $-F(\mathbf{q},\mathbf{v}_d)/ql$ as a function of $v_{d,\mathrm{eff}}/v_s$, for $v_{d,\mathrm{eff}}/v_s>1$, and several values of ql. Because $aF(\mathbf{q},\mathbf{v}_d)\gg 1$ at the resonance peaks, these curves are proportional to the peak amplification. Inspection of Fig. 6 shows that the peak amplification is a rather slowly varying monotonically increasing function of $v_{d,\mathrm{eff}}/v_s$ (provided that $v_{d,\mathrm{eff}}>v_s$).

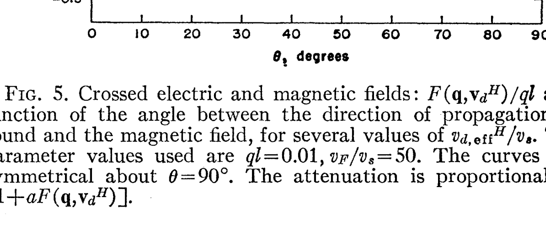

FIG. 5. Crossed electric and magnetic fields: $F(\mathbf{q},\mathbf{v}_d^H)/ql$ as a function of the angle between the direction of propagation of sound and the magnetic field, for several values of $v_{d,\mathrm{eff}}^H/v_s$. The parameter values used are $ql=0.01$, $v_F/v_s=50$. The curves are symmetrical about $\theta=90°$. The attenuation is proportional to $[1+aF(\mathbf{q},\mathbf{v}_d^H)]$.

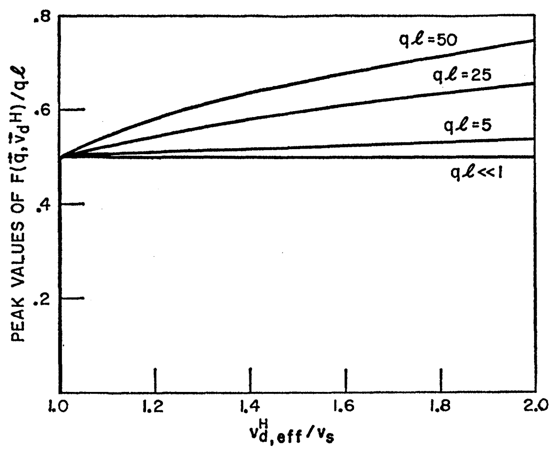

FIG. 6. Peak values of $-F(\mathbf{q},\mathbf{v}_d{}^H)/ql$ as a function of $v_{d,\mathrm{eff}}{}^H/v_s$ for several values of ql. The peak amplification is proportional to $-F(\mathbf{q},\mathbf{v}_d{}^H)$.

Turning now to the case $\mathbf{q}\perp\mathbf{H}$, it is found that

$$\alpha_j=\frac{2Nm}{\rho v_s\tau}[1+aF_\perp(\mathbf{q},\mathbf{v}_d)], \tag{4.6}$$

where

$$F_\perp(\mathbf{q},\mathbf{v}_d)=ql\frac{1}{6}\left(\frac{ql}{\omega_c\tau}\right)^2\left\{\frac{\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau+X_e}{(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau+X_e)^2+Y_e{}^2}+\frac{\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau+X_h}{(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau+X_h)^2+Y_h{}^2}\right\} \tag{4.7}$$

and

$$X_{e,h}+iY_{e,h}=\pm\frac{\lambda q_y v_d\tau}{\omega_c\tau(1-i\omega\tau)}+\frac{\lambda^2 q_x v_d\tau}{(\omega_c\tau)^2(1-i\omega\tau)}+i\frac{1}{3}\left(\frac{ql}{\omega_c\tau}\right)^2.$$

We note first that $F_\perp$ is proportional to $1/(\omega_c\tau)^2$ so that the amplification perpendicular to the magnetic field will usually be orders of magnitude smaller than that at an angle tilted towards the magnetic field. In fact, if $\omega_c\tau$ is large enough $|aF_\perp|<1$ everywhere except at resonance peaks, so that there will be no amplification except at these peaks.

The terms $X_{e,h}$ and $Y_{e,h}$ contribute only when $\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau$ is very small, of order $1/\omega_c\tau$. Therefore, in $X_{e,h}$ and $Y_{e,h}$ we may replace λ by 1. Also the second term in $X_{e,h}+iY_{e,h}$ is negligible in comparison with the third term if $v_d/v_F\ll ql$. This condition is satisfied at all but the lowest frequencies, so that we shall drop the second term. Then

$$X_{e,h}=\pm\frac{q_y v_d\tau}{\omega_c\tau[1+(\omega\tau)^2]},\qquad Y_{e,h}=\pm\frac{(q_y v_d\tau)(\omega\tau)}{\omega_c\tau[1+(\omega\tau)^2]}+\frac{1}{3}\left(\frac{ql}{\omega_c\tau}\right)^2. \tag{4.8}$$

For $q_y=0(\mathbf{q}\|\mathbf{v}_d)$ or for $\frac{1}{3}[1+(\omega\tau)^2](v_F/v_d)(ql/\omega_c\tau)\gg1$ (this second condition may be satisfied for large ql even though $ql\ll\omega_c\tau$):

$$\alpha_j=\frac{2Nm}{\rho v_s\tau}\left[1+aql\frac{1}{3}\left(\frac{ql}{\omega_c\tau}\right)^2\times\frac{\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau}{(\omega\tau-\mathbf{q}\cdot\mathbf{v}_d\tau)^2+[\frac{1}{3}(ql/\omega_c\tau)^2]^2}\right]. \tag{4.9}$$

This result may be compared with that of Dumke and Haering,[7] who calculated the amplification for the case $\mathbf{q}\|\mathbf{v}_d$. In the limit $\tau_r=\infty$ their formula is identical with the second term in the right-hand side of (4.9). The first term, $2Nm/\rho v_s\tau$, does not appear in their result. If we refer back to the derivation of the coefficient of attenuation (and to Ref. 14), we find that this term arises from the energy which is coherently returned to the sound wave at the average rate $(Nm/\tau)(\langle\mathbf{v}\rangle-\mathbf{u})\cdot\mathbf{u}$. This effect was not included in the work of Dumke and Haering, and consequently this term does not appear in their formula.

In the limiting cases under consideration, resonant amplification will occur, with the peak amplification at $\mathbf{q}\cdot\mathbf{v}_d\tau-\omega\tau=\frac{1}{3}(ql/\omega_c\tau)^2\approx0$. The value of $F_\perp(\mathbf{q},\mathbf{v}_d)$ at peak is $\frac{1}{2}ql$, independent of $\omega_c\tau$, and is, therefore, of the same order of magnitude as peak amplification for directions $\hat{q}$ tilted towards the magnetic field.

For $q_y\neq0$, in the limit $|q_y v_d\tau/\omega_c\tau(1-i\omega\tau)|\gg\frac{1}{3}(ql/\omega_c\tau)^2$ the situation is somewhat different. The amplification is a sum of two resonant functions [cf., Eq. (4.7)], but the peak value of $F_\perp(\mathbf{q},\mathbf{v}_d)$ is proportional to $1/\omega_c\tau$, so that in this limit, even the peak amplification is much smaller than amplification for directions of propagation tilted towards the magnetic field.

As we have seen, the peak value of $F(\mathbf{q},\mathbf{v}_d)/ql$ is not sensitively dependent upon frequency. In fact, for $ql\ll1$ and for $\mathbf{q}\perp\mathbf{H}$, it is independent of frequency; for $ql\gg1$ it increases slowly with frequency (see Fig. 6). Therefore, to a good approximation, the peak amplification varies with frequency as ql.

In conclusion, for each value of v_d, such that $v_d>v_s$, amplification occurs for a certain set of directions of propagation $\hat{q}$; the peak amplification is not very strongly dependent upon the value of v_d, and varies with frequency as ql. The directions of propagation for which resonant amplification occur are strongly dependent upon ql: For $ql\gg1$ these directions correspond to $\hat{q}\cdot\mathbf{v}_d-v_F(\hat{q}\cdot\hat{H})\approx v_s$, and for $ql\ll1$ they correspond to $\hat{q}\cdot\mathbf{v}_d\approx v_s$. Hence, the optimum conditions for observation of amplification are high ql, and directions of propagation tilted slightly from the perpendicular to the magnetic field.

The fact that the peak amplification is rather insensitive to the value of the drift velocity has consequences with respect to the anomalous resistance experiments discussed in the introduction. From the calculations of amplification for the case $\mathbf{v}_d \| \mathbf{q}$, it appeared that the amplification approaches zero asymptotically as v_d increases. If this were true for the peak amplification, then no acoustic waves would be present in the medium for large values of v_d, and the resistance would return to its value for $v_d < v_s$. However, anomalous resistance *was* observed at values of the drift velocity for which the amplification in the forward direction is already quite small. From the results given above, it is clear that this is because the peak amplification is in fact insensitive to v_d, and does *not* decrease as v_d increases, so that acoustic waves are present even at large values of the drift velocity.

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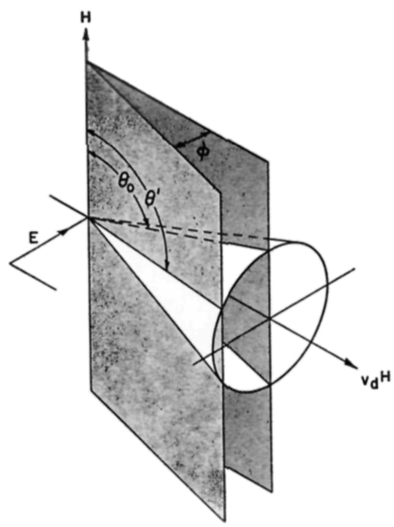

FIG. 3. Geometry for crossed field case. Amplification occurs for all directions of propagation $\hat{q}$ within the cone, which is defined by $\hat{q}\cdot \mathbf{v}_d{}^H = v_s$.